Quantum information theory of entanglement and measurement ⋆

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We present a quantum information theory that allows for a consistent description of entanglement. It parallels classical (Shannon) information theory but is based entirely on density matrices (rather than probability distributions) for the description of quantum ensembles. We find that quantum conditional entropies can be negative for entangled systems, which leads to a violation of well-known bounds in Shannon information theory. Such a unified information-theoretic description of classical correlation and quantum entanglement clarifies the link between them: the latter can be viewed as "super-correlation" which can induce classical correlation when considering a tripartite or larger system. Furthermore, negative entropy and the associated clarification of entanglement paves the way to a natural information-theoretic description of the measurement process. This model, while unitary and causal, implies the well-known probabilistic results of conventional quantum mechanics. It also results in a simple interpretation of the Kholevo theorem limiting the accessible information in a quantum measurement.

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1 Introduction

The recent vigorous activity in the fields of quantum information processing (quantum computation) and quantum communication (quantum cryptography, teleportation, and superdense coding) has necessitated a better understanding of the relationship between classical and quantum variables (see, *e.g.*, [3,10,11,17]). In classical physics, information processing and communication is best described by Shannon information theory [23], which succinctly associates *information* with randomness *shared* by two physical ensembles. Quantum information theory on the other hand is concerned with quantum bits (qubits) rather than bits, and the former obey quantum laws quite different from the classical physics of bits that we are used to [21]. Most importantly, qubits can exist in quantum *superpositions*, a notion essentially foreign to classical mechanics, or even classical thinking. To accommodate the relative phases in quantum superpositions, quantum information theory must be based on mathematical constructions which reflect these: the quantum mechanical density matrices. The central object of information theory, the entropy, has been introduced in quantum mechanics by von Neumann [24]

$$S(\rho) = -\mathrm{Tr}\,\rho\log\rho\ . \tag{1}$$

Its relationship to the Boltzmann-Gibbs-Shannon entropy

$$H(p) = -\sum_i p_i \log p_i \tag{2}$$

becomes obvious when considering the von Neumann entropy of a mixture of orthogonal states. In this case, the density matrix ρ in (1) contains classical probabilities p_i on its diagonal, and $S(\rho) = H(p)$. In general, however, quantum mechanical density matrices have off-diagonal terms, which, for pure states, reflect the relative quantum phases in superpositions.

In classical statistical physics, the concept of conditional and mutual probabilities has given rise to the definition of conditional and mutual entropies. These can be used to elegantly describe the trade-off between entropy and information in measurement, as well as the characteristics of a transmission channel. For example, for two statistical ensembles A and B, the measurement of (variables of) A by B is expressed by the equation for the entropies

$$H(A) = H(A|B) + H(A{:}B)\ . \tag{3}$$

Here, $H(A|B)$ is the entropy of A after having measured those pieces that become correlated in B [thereby apparently reducing $H(A)$ to $H(A|B)$], while

$H(A{:}B)$ is the *information* gained about A via the measurement of B. As is well-known, $H(A|B)$ and $H(A{:}B)$ compensate each other such that $H(A)$ is unchanged, ensuring that the second law of thermodynamics is not violated in a measurement in spite of the decrease of $H(A|B)$ [16]. Mathematically, $H(A|B)$ is a *conditional* entropy, and is defined using the conditional probability $p_{i|j}$ and the joint probability p_{ij} describing random variables from ensembles A and B:

$$H(A|B) = -\sum_{ij} p_{ij} \log p_{i|j} \ . \tag{4}$$

The information or mutual entropy (or correlation entropy) $H(A{:}B)$, on the other hand is defined via the mutual probability $p_{i:j} = p_i \, p_j / p_{ij}$ as

$$H(A{:}B) = -\sum_{ij} p_{ij} \log p_{i:j} \ . \tag{5}$$

Simple relations such as $p_{ij} = p_{i|j} \, p_j$ imply equations such as (3) and all the other usual relations of classical information theory. Curiously, a quantum information theory paralleling these constructions has never been attempted. Rather, a "hybrid" procedure is used in which quantum probabilities are inserted in the classical formulae of Shannon theory, thereby losing the quantum phase crucial to density matrices (see, *e.g.*, [29]). Below, in Section 2, we show that a consistent quantum information theory can be developed that parallels the construction outlined above, while based entirely on matrices [6]. This formalism allows for a proper *information-theoretic* description of quantum entanglement, unified with the standard description of classical correlations, as shown in Section 3. As a result, most of the classical concepts involving entropies for composite systems in Shannon theory can be extended to the quantum regime, and this provides a simple intuitive framework for dealing with quantum entropies. In the fourth section, we analyze quantum measurement in this information-theoretic language and point out how this picture leads to a unitary and causal view of quantum measurement devoid of wave function collapse [7]. In Section 5, we analyze Bell-type measurements in terms of information, as an application of this model. In Section 6, we conclude by considering a simple quantum information-theoretic derivation of the Kholevo theorem (which limits the amount of information that can be extracted in a measurement).

2 Quantum information theory

Let us consider the information-theoretic description of a bipartite quantum system AB. A straightforward quantum generalization of Eq. (4) suggests the

definition

$$S(A|B) = -\mathrm{Tr}_{AB}[\rho_{AB} \log \rho_{A|B}] \tag{6}$$

for the quantum conditional entropy. In order for such an expression to hold, we need to define the concept of a "conditional" density matrix,

$$\rho_{A|B} = \lim_{n\to\infty} \left[\rho_{AB}^{1/n}(\mathbf{1}_A \otimes \rho_B)^{-1/n}\right]^n \ , \tag{7}$$

which is the analogue of the conditional probability $p_{i|j}$. Here, $\mathbf{1}_A$ is the unit matrix in the Hilbert space for A, $\otimes$ stands for the tensor product in the joint Hilbert space, and

$$\rho_B = \mathrm{Tr}_A[\rho_{AB}] \tag{8}$$

denotes a "marginal" or reduced density matrix, analogous to the marginal probability $p_j = \sum_i p_{ij}$. The symmetrized product involving the infinite limit in the definition of the conditional density matrix (7) is a technical requirement due to the fact that joint and marginal density matrices do not commute in general. This definition for $\rho_{A|B}$ implies that the standard relation

$$S(A|B) = S(AB) - S(B) \tag{9}$$

holds for the quantum entropies and that $S(A|B)$ is invariant under any unitary transformation of the product form $U_A \otimes U_B$. More precisely, the conditional density matrix $\rho_{A|B}$ as defined by Eq. (7) is a *positive* Hermitian operator in the joint Hilbert space, whose spectrum is invariant under $U_A \otimes U_B$. However, in spite of the apparent similarity between the quantum definition for $S(A|B)$ and the standard classical one for $H(A|B)$, dealing with matrices (rather than scalars) opens up a quantum realm for information theory exceeding the classical one. The crucial point is that, while $p_{i|j}$ is a probability distribution in i (i.e., $0 \leq p_{i|j} \leq 1$), its quantum analogue $\rho_{A|B}$ is *not* a density matrix: while Hermitian and positive, it can have eigenvalues *larger* than one, and, consequently, the associated conditional entropy $S(A|B)$ can be *negative*. Only such a *matrix-based* formalism consistently accounts for the well-known non-monotonicity of quantum entropies (see, *e.g.*, [25]). In other words, $S(A|B) < 0$ means that it is acceptable, in quantum information theory, to have $S(AB) < S(B)$, *i.e.*, the entropy of the entire system AB can be smaller than the entropy of one of its subparts B, a situation which is of course forbidden in classical information theory. This happens for example in the case of quantum *entanglement* between A and B, as will be illustrated below [6].

The "non-classical" spectrum of the conditional density matrix $\rho_{A|B}$ is related to the question of the separability of the mixed state ρ_{AB}. First, the concavity of $S(A|B)$, a property related to strong subadditivity of quantum entropies [25], implies that any separable state

$$\rho_{AB} = \sum_k w_k \, \rho_A^{(k)} \otimes \rho_B^{(k)} \qquad\qquad (\text{with } \sum_k w_k = 1) \tag{10}$$

is associated with a non-negative $S(A|B)$. (Note that the converse is not true.) Indeed, each product component $\rho_A^{(k)} \otimes \rho_B^{(k)}$ of a separable state is associated with the conditional density matrix

$$\rho_{A|B}^{(k)} = \rho_A^{(k)} \otimes \mathbf{1}_B \tag{11}$$

so that we have

$$S(A|B) \geq \sum_k w_k S(\rho_A^{(k)}) \geq 0 \; . \tag{12}$$

This shows that the non-negativity of conditional entropies is a *necessary* condition for separability. This condition can be shown to be equivalent to the non-violation of entropic Bell inequalities [8]. Secondly, it is easy to check from Eq. (6) that, if $S(A|B)$ is negative, $\rho_{A|B}$ must admit at least one "non-classical" eigenvalue (*i.e.*, an eigenvalue exceeding one), while the converse again does not hold. This results from the fact that $\mathrm{Tr}(\rho\sigma) \geq 0$ if ρ and σ are positive Hermitian matrices. This suggests the conjecture that a strong necessary condition for separability is that *all* the eigenvalues of $\rho_{A|B}$ are "classical" (≤ 1). When applied to the case of a Werner state (an impure singlet state), this separability condition turns out to be necessary (and sufficient for a 2×2 Hilbert space) [6], since it reduces exactly to the condition derived in Ref. [19] by considering the positivity of the partial transpose of ρ_{AB}. When applied to a randomly generated mixture of product states, this condition is always fulfilled (*i.e.*, all the eigenvalues of $\rho_{A|B}$ and $\rho_{B|A}$ are ≤ 1). This opens the possibility that it could be a stronger necessary (or perhaps even sufficient) condition for separability in a Hilbert space of arbitrary dimensions. Further work will be devoted to this question.

Similarly to what we have done for the conditional entropy, the quantum analogue of the mutual entropy can be constructed by defining a "mutual" density matrix

$$\rho_{A:B} = \lim_{n\to\infty} \left[(\rho_A \otimes \rho_B)^{1/n} \rho_{AB}^{-1/n} \right]^n \;\; , \tag{13}$$

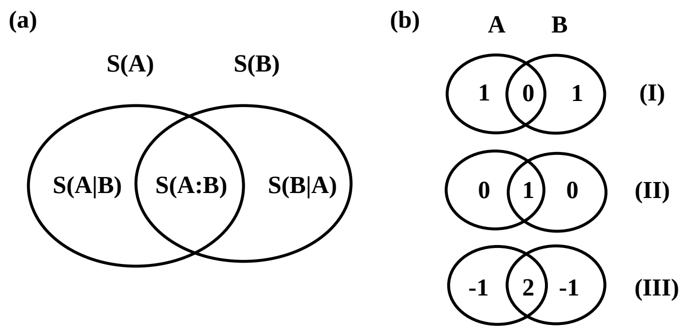
the analogue of the mutual probability $p_{i:j}$. As previously, this definition implies the standard relation

$$S(A{:}B) = S(A) - S(A|B) = S(A) + S(B) - S(AB) \tag{14}$$

between the quantum entropies. This definition extends the classical notion of *mutual information* or correlation entropy $H(A{:}B)$ to the quantum notion of *mutual entanglement* $S(A{:}B)$. Note that all the above quantum definitions reduce to the classical ones for a diagonal ρ_{AB}, which suggests that Eqs. (7) and (13) are reasonable assumptions. (It is possible that other definitions of $\rho_{A|B}$ and $\rho_{A:B}$ can be proposed, but we believe this choice is simplest.) The proposed matrix-based information theory therefore includes Shannon theory as a special case, while it describes quantum entanglement as well. Since the definition of mutual entanglement $S(A{:}B)$ covers classical correlations also, $S(A{:}B)$ must be considered as a general measure of correlations *and* "super-correlations" in information theory, which applies to pure as well as mixed states. It is worth noticing that this does *not* mean that the mutual entanglement characterizes the purely quantum correlation between A and B (that part which can be purified to singlet states); rather $S(A{:}B)$ does not separate correlation and entanglement, it is a measure of both. In this sense, $S(A{:}B)$ differs from various definitions of the entropy of entanglement which can be found in the literature [4]. Finally, we show in Ref. [1] that, besides being the proper quantum counterpart of correlation entropy, $S(A{:}B)$ also turns out to be a basic quantity in the search for a quantum counterpart of Shannon's fundamental coding theorem on noisy channels. Indeed, our proposed definition for the capacity for entanglement transmission through a quantum channel is written as the maximum achievable mutual entanglement $S(A{:}B)$, in analogy with the classical definition.

As we shall see in the next section, our quantum matrix-based formalism can be successfully applied to the quantum entanglement of more than two systems by extending the various classical entropies that are defined in the Shannon information-theoretic treatment of a multipartite system. This accounts for example for the creation of classical correlation through quantum entanglement in a tripartite (or larger) system. Also, the quantum analogue of all the fundamental relations between classical entropies (such as the chain rules for entropies and mutual entropies) holds in quantum information theory and have the same intuitive interpretation, and we make extensive use of it in [1,7,8]. Let us close this section by suggesting a simple diagrammatic way of representing quantum entropies which provides intuitive insight into this information-theoretic description of entanglement. In the case of a bipartite system, the relations between $S(A)$, $S(B)$, $S(AB)$, $S(A|B)$, $S(B|A)$, and $S(A{:}B)$ are conveniently summarized by a Venn-like entropy diagram, as shown in Fig. 1a. The important difference between classical and quan-

Fig. 1. (a) General entropy diagram for a quantum bipartite system AB. (b) Entropy diagrams for three cases of a system of 2 qubits: (I) independent, (II) classically correlated, (III) quantum entangled.



tum entropy diagrams is that the basic inequalities relating the entropies are "weaker" in the quantum case, allowing for negative conditional entropies and "excessive" mutual entropies [6]. For example, the upper bound for the mutual entropy (which is directly related to the channel capacity) is

$$H(A{:}B) \leq \min[H(A), H(B)] \tag{15}$$

in classical information theory, as a consequence of the inequality $H(AB) \geq \max[H(A), H(B)]$, while it is

$$S(A{:}B) \leq 2\min[S(A), S(B)] \tag{16}$$

in quantum information theory, as a result of the Araki-Lieb inequality [25] $S(AB) \geq |S(A){-}S(B)|$. This means that a quantum channel has a capacity for entanglement transmission that can reach twice the classical upper bound [1,6]; this is apparent for instance in the superdense coding scheme [5].

3 Correlation versus entanglement and multipartite systems

We show in Fig. 1b the entropy diagram corresponding to three limiting cases of a bipartite system of two dichotomic variables (e.g., 2 qubits): independent variables (case I), classically correlated variables (case II), and quantum entangled variables (case III). In all three cases, each subsystem taken separately is in a mixed state of entropy $S(A) = S(B) = 1$ bit. Cases I and II correspond to classical situations (which can of course be described in our matrix-based formalism as well, using diagonal matrices), while case III is a purely quantum situation which violates the bounds of classical information theory [6]. Let us focus on case III, since cases I and II are standard. This case corresponds to

an EPR pair [1], characterized by the pure state

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \ , \tag{17}$$

and, accordingly, it is associated with a vanishing total entropy $S(AB) = 0$. Using the density matrix of the joint system $\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$, we see that subpart A (or B) has the marginal density matrix

$$\rho_A = \mathrm{Tr}_B[\rho_{AB}] = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \ , \tag{18}$$

and is therefore in a mixed state of positive entropy. This purely quantum situation corresponds to the unusual entropy diagram (–1,2,–1) shown in Fig. 1b. That the EPR situation cannot be described classically is immediately apparent when considering the associated density matrices. The joint and the marginal density matrices can be written in basis $\{00, 01, 10, 11\}$ as

$$\rho_{AB} = \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix} , \qquad \rho_A = \rho_B = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} . \tag{19}$$

so that we obtain for the conditional density matrix [2]

$$\rho_{A|B} = \rho_{AB}(\mathbf{1}_A \otimes \rho_B)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} . \tag{20}$$

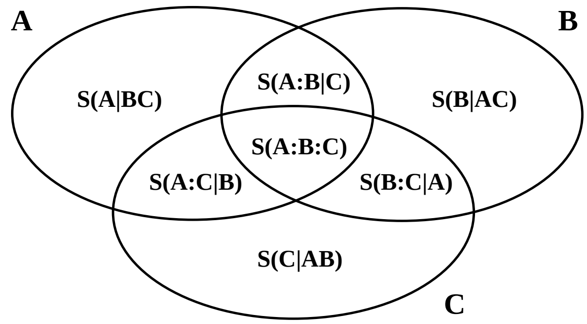
Plugging (19) and (20) into definition (6) immediately yields $S(A|B) = -1$, which results in

$$S(AB) = S(A) + S(B|A) = 1 - 1 = 0 \tag{21}$$

[1] Although we use the term "EPR state" for the wave-function (17), this state is in fact one of the *Bell* states, which are a generalization of the EPR singlet state.

[2] Note that for EPR pairs, joint and marginal density matrices commute, simplifying definitions (7) and (13).

Fig. 2. Ternary entropy Venn-diagram for a general tripartite system ABC. The component entropies are defined in the text.



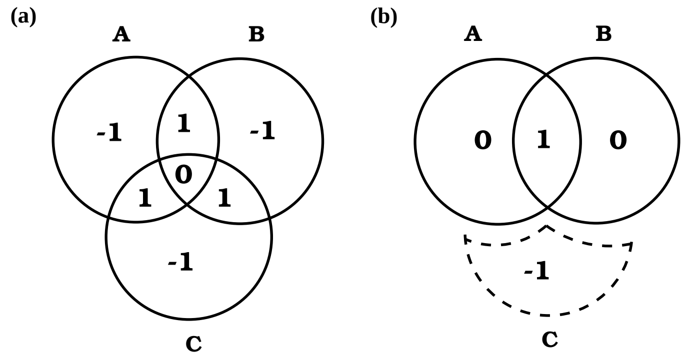
as expected. This is a direct consequence of the fact that $\rho_{A|B}$ has one "non-classical" (> 1) eigenvalue, 2. It is thus misleading to describe an EPR-pair (or any of the Bell states) as a correlated state within Shannon information theory, since negative conditional entropies are crucial to its description [3]. Still, classical *correlations* [with entropy diagram $(0, 1, 0)$] emerge when *observing* an entangled EPR pair. Indeed, after measuring A, the outcome of the measurement of B is known with 100% certainty. The key to this discrepancy lies in the information-theoretic description of the measurement process [7]. Anticipating the next section, let us just mention that the *observation* of quantum entangled states such as an EPR pair gives rise to classical correlations between the two measurement *devices* while keeping the entanglement between the two parties (particle + measurement device) unchanged, thereby creating the confusion between entanglement and correlation.

More generally, the concept of negative conditional entropy turns out to be very useful to describe *multipartite* quantum systems, and it gives new insight into the creation of classical correlations from quantum entanglement. In the case of a tripartite system, the quantum entropies involved can be represented by a Venn-like diagram, as shown in Fig. 2. The conditional entropies $S(A|BC)$, $S(B|AC)$, and $S(C|AB)$ are a straightforward generalization of conditional entropies in a bipartite system, *i.e.*, $S(A|BC) = S(ABC) - S(BC)$, etc. The entropies $S(A{:}B|C)$, $S(A{:}C|B)$, and $S(B{:}C|A)$ correspond to conditional mutual entropies. They characterize the mutual entanglement between two of the subsystems when the third is known. In perfect analogy with the classical definition, one can write, for example,

$$S(A{:}B|C) = S(A|C) - S(A|BC) \; . \tag{22}$$

[3] In Ref. [6], we suggest that EPR pairs are better understood in terms of a qubit-antiqubit pair, where the qubit (antiqubit) carries plus (minus) one bit, and where antiqubits are interpreted as qubits traveling *backwards* in time.

Fig. 3. (a) Ternary entropy diagram for an “EPR-triplet” or GHZ state. (b) Entropy diagram for subsystem AB unconditional on C.



This is a straightforward generalization of Eq. (14) where all the entropies are conditional on C. A trivial calculation gives also the expression of the conditional mutual entropy in terms of total entropies

$$S(A{:}B|C) = S(AC) + S(BC) - S(C) - S(ABC) \tag{23}$$

This last expression illustrates that the conditional mutual entropies are always non-negative as a consequence of strong subadditivity of quantum entropies (see, *e.g.*, [25]), a property that will be useful in the following. The entropy in the center of the diagram is a *ternary* mutual entropy, defined as

$$S(A{:}B{:}C) = S(A{:}B) - S(A{:}B|C) \tag{24}$$

(this generalizes Eq. (14) for a mutual entropy rather than a total entropy). Using Eq. (23), this can be written in a more symmetric way as

$$\begin{aligned} S(A{:}B{:}C) = {} & S(A) + S(B) + S(C) - S(AB) - S(AC) - S(BC) \\ & + S(ABC)\ . \end{aligned} \tag{25}$$

More generally, relations between entropies in a multipartite system can be written, such as the “chain rules” for quantum entropies

$$S(A_1 \cdots A_n) = S(A_1) + S(A_2|A_1) + S(A_3|A_1 A_2) + \cdots \tag{26}$$

or for quantum mutual entropies

$$S(A_1 \cdots A_n{:}B) = S(A_1{:}B) + S(A_2{:}B|A_1) + S(A_3{:}B|A_1 A_2) + \cdots \tag{27}$$

Let us consider as an illustration a tripartite system ABC in a Greenberger-Horne-Zeilinger (GHZ) [13] state [4],

$$|\psi_{ABC}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \ . \tag{28}$$

As it is a pure (entangled) state, the total entropy is $S(ABC) = 0$. The corresponding ternary entropy diagram of ABC is shown in Fig. 3a. Note that the vanishing *ternary* mutual entropy

$$S(A{:}B{:}C) = 0 \tag{29}$$

in the center of the diagram is generic to any entangled tripartite system in a pure state [7] [5]. Indeed, the Schmidt decomposition of the pure state $|\psi_{ABC}\rangle$ implies that $S(AB) = S(C)$, $S(AC) = S(B)$, and $S(BC) = S(A)$. This feature will be important in the following section as it implies that no information (in the sense of Shannon theory) is extracted in the measurement of a pure state. Fig. 3a shows clearly that, grouping say A and B (considering them as a single entity) results in entanglement [diagram (-1,2,-1)] between AB and C. On the other hand, when tracing over the degree of freedom associated with C, say, the resulting marginal density matrix for subsystem AB is

$$\rho_{AB} = \mathrm{Tr}_C[\rho_{ABC}] = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|) \ , \tag{30}$$

corresponding to a classically correlated system [diagram (0,1,0)]. As the density matrix *fully* characterizes a quantum system, subsystem AB (unconditional on C, i.e., ignoring the existence of C) is in this case physically *indistinguishable* from a statistical ensemble prepared with an equal number of $|00\rangle$ and $|11\rangle$ states. Thus, A and B are correlated in the sense of Shannon theory if C is ignored. The "tracing over" operation depicted in Fig. 3b illustrates this creation of classical correlation from quantum entanglement. In short, the EPR-triplet entails quantum entanglement between any part, e.g. C, and the rest of the system AB. The subsystem AB *unconditional* on C has a positive entropy $S(AB)$ of 1 bit, and is indistinguishable from a classical correlated mixture. On the other hand, the entropy of C conditional on AB, $S(C|AB)$, is negative and equal to -1 bit, thereby counterbalancing $S(AB)$ to yield a vanishing combined entropy

$$S(ABC) = S(AB) + S(C|AB) = 0 \ , \tag{31}$$

[4] The GHZ state can also be viewed as an "EPR-triplet", a generalization of an EPR-pair to three parties.

[5] For a multipartite system, the mutual entropy between the n parts is equal to $1 + (-1)^n$.

as expected in view of the entanglement between AB and C. The above can be extended in a straightforward manner to composite systems, and this will be central to the measurement process.

4 Quantum measurement

According to von Neumann [24], a consistent description of the measurement process must involve the interaction between the observed quantum system and a *quantum* measurement device. Such a view is in contrast with the Copenhagen interpretation of quantum mechanics (see, e.g., [26]) stating that the measurement is the non-causal process of projecting the wave-function, which results from the interaction with a *classical* apparatus. A classical apparatus is defined as one where the "pointer" variables take on *definite* values, and which therefore cannot reflect quantum superpositions. For 70 years, the Copenhagen interpretation has never failed in predicting a single experimental fact, which certainly has helped in cementing its reputation [26]. On the other hand, if the foundations of quantum mechanics are believed to be solid, it cannot be denied that measurement is *not* an abstract non-causal operation acting on wave functions, but rather a genuine interaction between two physical *quantum* systems: the observed system Q and the measurement device, or the ancilla A. This is the essence of the von Neumann theory of measurement.

Assume then that a quantum system is initially in state

$$|Q\rangle = \sum_i \alpha_i |a_i\rangle \tag{32}$$

expressed in the basis $\{|a_i\rangle\}$ of eigenvectors of an arbitrary observable (the one that we are measuring). Then, the von Neumann measurement is described by the *unitary* transformation that evolves the initial state of the joint system $|Q, A\rangle = |Q, 0\rangle$ into the state

$$|QA\rangle = \sum_i \alpha_i |a_i, i\rangle \tag{33}$$

with $\{|i\rangle\}$ denoting the eigenstates of the ancilla ($|0\rangle$ is the reference initial state of the ancilla). Such a transformation was interpreted by von Neumann as inducing *correlations* between the system Q and the ancilla A. Indeed, if $|Q\rangle$ is initially in one of the eigenstates $|a_i\rangle$ (i.e., if it is *not* in a superposition), the "pointer" in A that previously pointed to zero now points to the eigenvector $|i\rangle$ which labels outcome i, suggesting that a measurement has been performed.

Now, a basic problem occurs if the initial state of Q is a superposition, as

in Eq. (32), that is, if Q is *not* in an eigenstate of the considered observable. Then, according to Eq. (33), the apparatus apparently points to a *superposition* of i's, a fact which obviously contradicts our everyday-life experience. In classical physics, a variable has, at any time, a definite value that can be recorded. Experiments show that a quantum measurement is probabilistic in nature, that is *one* of the possible outcomes (drawn from a probability distribution) becomes *factual*. In other words, a quantum superposition evolves into a mixed state. This apparent necessity led von Neumann to introduce an *ad hoc*, non-unitary, *second* stage of the measurement, called *observation*. In this stage, the measurement is "observed", and a collapse occurs in order to yield a classical result from a quantum superposition. The central point in the quantum information-theoretic interpretation of the measurement problem presented below (see also Ref. [7]) is that, in general, the state described by Eq. (33) is *entangled*, not just correlated. As emphasized earlier, entangled states have an information-theoretic description distinct from correlated states, which provides them with very peculiar properties. For example, it has been shown that an arbitrary quantum state cannot be *cloned* [27] precisely because of the entanglement between the system Q and the ancilla A. If the system is in a state belonging to a set of *orthogonal* states, on the other hand, a faithful copy of the quantum state can be obtained applying a von Neumann measurement. As a consequence it appears that an *arbitrary* state (one which is *not* one of the eigenstates of the observable considered) can not be *measured* without creating entanglement.

Let us show that unitary evolution [such as the one giving rise to Eq. (33)] can be reconciled with the creation of randomness in the measurement process if it is recognized that the creation of entanglement (rather than correlation) is *generic* to a quantum measurement, *and* if this entanglement is properly described in quantum information theory using the concept of negative entropy (see also Ref. [7]). This reconciliation is brought about by a redescription of the second stage of measurement, the observation, without involving an irreversible loss of information to a macroscopic environment. In that respect, our model is distinct from the environment-induced decoherence model, one of the prevalent contemporary views of quantum measurement (see *e.g.* [30]). In order to observe the measurement, a system involving generally a large number of degrees of freedom has to interact with Q. In Eq. (33), Q has interacted with a single degree of freedom of the ancilla A (first stage of the measurement), which led to an entangled state. As emphasized before, the creation of an entangled state does not mean that a measurement has been performed, since our (classical) perception of a measurement is intrinsically related to the existence of (classical) correlations. In order for classical correlations to emerge, a third degree of freedom (another ancilla A') has to be involved. Now, iterating the von Neumann measurement, A' interacts with AQ so that

the resulting state of the combined system is

$$|QAA'\rangle = \sum_i \alpha_i |a_i, i, i\rangle \ , \tag{34}$$

where the eigenstates of A' are also denoted by $|i\rangle$ for simplicity. The state so created is pure $[S(QAA') = 0]$, akin to an "EPR-triplet" since the system has undergone only unitary transformations from a pure initial state $|Q, 0, 0\rangle$. The point is that, considering the state of the entire ancilla AA' *unconditionally* on system Q yields a *mixed state*

$$\rho_{AA'} = \mathrm{Tr}_Q[\rho_{QAA'}] = \sum_i |\alpha_i|^2 |i, i\rangle\langle i, i| \tag{35}$$

describing maximal correlation between A and A', that is

$$S(A{:}A') = S(A) = S(A') = S(AA') \ . \tag{36}$$

The second stage consists in observing *this* classical correlation (that extends, in practice, to the 10^{23} particles which constitute the macroscopic measurement device). Note that a macroscopic measurement device is not required here, since only two ancillary degrees of freedom A and A' are enough to generate correlation in the tripartite entangled system QAA'. The entropy diagram characterizing QAA' is of the same kind as the one depicted in Fig. 3, but filled in with a constant different from 1 in general. Paradoxically, it is the physical state of the *ancilla* which contains the outcome of the measurement, whereas the quantum state Q itself must be *ignored* to observe correlations. This crucial point is easily overlooked, since intuition dictates that performing a measurement means somehow "observing the state of Q". Rather, a measurement is constructed such as to *infer* the state of Q from that of the ancilla—but ignoring Q itself. The correlations (in AA') which emerge from the fact that a part (Q) of an entangled state (QAA') is ignored give rise to the classical idea of a measurement. This view of the measurement process insists only on the "self-consistency" of the measurement device, while abandoning the 100% correlation between the latter and the quantum system Q, a cornerstone of decoherence models. More precisely, no information (in the sense of Shannon theory) about Q is obtained from the ancilla. Indeed, using Eqs. (23) and (36), we have

$$S(Q{:}A{:}A') = S(A{:}A') - S(A{:}A'|Q) = 0 \tag{37}$$

meaning that the mutual entropy between A and A' (the observed correlation) is *not* shared with Q. This is a consequence of the fact that Q is initially in a pure state. We will see in the next section that information (Shannon mutual

entropy) can only be acquired in the situation where a *mixed* state is measured. After measurement, the quantum entropy of the ancilla (unconditional on Q)

$$S(AA') = H[p_i] \qquad \text{with } p_i = |\alpha_i|^2 \tag{38}$$

is interpreted as the "physical" entropy of Q. This happens to be the classical entropy associated with the probability distribution of the random outcomes, $p_i = |\alpha_i|^2$, that is the probabilities predicted by quantum mechanics. Thus the unconditional entropy of the ancilla is equal to the entropy of Q predicted in "orthodox" quantum mechanics (which involves the projection of the wave function). Still, the entropy of Q conditional on AA' is *negative*, and exactly compensates $S(AA')$ to allow for a vanishing entropy for the joint system,

$$S(AA') + S(Q|AA') = S(QAA') = 0 \tag{39}$$

This then emphasizes how measurement can be probabilistic in nature, while at the same time being described by a unitary process (which does *not* permit the evolution of pure into mixed states).

The appearance of a wave-function collapse, crucial in the physics of sequential measurements, can also be interpreted in this information-theoretic picture. If a second ancilla B (in general, also a large number of degrees of freedom) interacts with Q in order to measure the *same* observable (after a first measurement involving ancilla A), the result is an "EPR-$n$plet" (consisting of all the degrees of freedom of A, B, and the measured quantum state Q). To simplify, let us consider two ancillary variables A and B (and neglect their amplification). Then, the final quantum state after the sequential measurement is

$$|QAB\rangle = \sum_i \alpha_i |a_i, i, i\rangle \tag{40}$$

illustrating clearly that the state of A and B (unconditional on Q) are *classically* maximally correlated just as described earlier. This is the basic consistency requirement for two consecutive measurements of the same variable: we must have $S(B|A) = 0$. The standard assertion of orthodox quantum mechanics is that, after the first measurement, the wave function of Q is projected on $|a_i\rangle$, the observed eigenstate, so that any following measurement yields the same value i without any remaining uncertainty since the state of Q is now $|a_i\rangle$. As we just showed, such a classical correlation between the *outcome* of two measurements actually involves *no* collapse; rather, the vanishing remaining uncertainty of the second measurement [reflected by the vanishing conditional entropy $S(B|A) = 0$] is due to the fact that one considers only part of an entangled system.

More interestingly, in the case where the ancilla B measures another observable, Eq. (40) becomes then

$$|QAB\rangle = \sum_{i,j} \alpha_i U_{ij} |b_j, i, j\rangle \tag{41}$$

where $\{|b_j\rangle\}$ are the eigenvectors of the second observable, $U_{ij} = \langle b_j|a_i\rangle$, and $\{|j\rangle\}$ denote the eigenstates of the second ancilla B. The resulting entropy diagram for AB (obtained by tracing over Q) gives rise to an *entropic* uncertainty relation [7]

$$S(A) + S(B) \geq \min_i H\left[|U_{ij}|^2\right]_{i \text{ fixed}} \tag{42}$$

where the right-hand side stands for the Shannon entropy of the probability distribution resulting from the expansion of the eigenvector $|a_i\rangle$ of the first observable into the eigenbasis of the second observable, minimized over i. This reflects the fact that the sequential measurement of two non-commuting observables (such that U_{ij} is not the identity or a permutation matrix) must generate a non-zero entropy. Eq. (42) is compatible with the uncertainty relations found in the literature (see, *e.g.*, [9]), but can be shown to be stronger in intermediate situations between compatible and complementary (maximally incompatible) observables [7].

5 Bell-type measurements

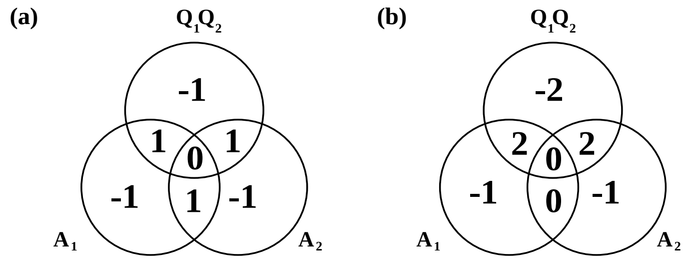
In order to illustrate the information-theoretic analysis of measurement described above, let us consider the measurement of an EPR pair. This should also clarify how quantum entanglement can have the appearance of classical correlation in such an experiment. Let us prepare a bipartite system Q_1Q_2 in the EPR-entangled state

$$|Q_1Q_2\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle) \tag{43}$$

and separate the two members at remote locations in space. At each location, the system (Q_1 or Q_2) is measured by interacting with an ancilla (A_1 or A_2), following the same procedure as before. In brief, each system (Q_1 or Q_2) becomes entangled with its corresponding ancilla, resulting in the entangled state

$$|Q_1Q_2A_1A_2\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow 11\rangle + |\downarrow\downarrow 00\rangle) \tag{44}$$

Fig. 4. Ternary entropy diagram for the measurement of an EPR pair. (a) A_1 and A_2 both measure the spin z-component σ_z. (b) A_1 measures σ_z while A_2 measures σ_x.



for the entire system. Note that an ancilla in state $|1\rangle$ means that a spin-up has been measured, and conversely. (Obviously, this corresponds to the measurement of the spin-projection along the z-axis; the measurement of different spin-components of Q_1 and Q_2 can be considered along the same lines.) As previously, we describe the ancilla with just one internal variable, even though in practice it must be thought of as consisting of a large number of correlated ones. The important point here is that, despite the fact that Q_1 and Q_2 were initially in an entangled state [characterized by the $(-1, 2, -1)$ entropy diagram], the state of the two ancillae unconditional on Q_1 and Q_2 is a mixed (classically correlated) state

$$\rho_{A_1A_2} = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|) \tag{45}$$

Thus, the ancillae are *correlated*: the corresponding entropy diagram $(0, 1, 0)$ clearly shows that, after observing A_1, for instance, the state of A_2 can be inferred without any uncertainty, *i.e.*, $S(A_2|A_1) = 0$. However, this must *not* be attributed to the existence of classical correlation between Q_1 and Q_2; rather it is the act of measuring which gives rise to this appearance. The entropy relations between Q_1, Q_2, A_1 and A_2 can be summarized by an entropy diagram (Fig. 4a). This emphasizes that it is the same mechanism which is at the origin of the coincidence between the observed spin-projection for both particles in an EPR experiment and at the core of the consistency between sequential measurements on a single quantum system. [6] In the former case, the mechanism is well known and accepted, while in the latter case, it is more difficult to discern and an *ad hoc* collapse is therefore often wrongly invoked. For completeness, the entropy diagram describing the situation where the ancillae A_1 and A_2 measure *orthogonal* spin projections (*e.g.*, σ_x and σ_z) is shown in Fig. 4b. When tracing over Q_1 and Q_2, it is obvious that the ancillae A_1 and A_2 are statistically independent, which accounts for the fact that two apparently independent random variables (σ_x and σ_z) are measured. In reality, the entire system is entangled in a particular way: A_1 and A_2 are entangled *separately*

[6] We thank Zac Walton for pointing this out to us.

with Q_1Q_2. Note finally that the violation of Bell inequalities occurring in the measurement of EPR pairs can also be analyzed from an information-theoretic point of view, as shown in Ref. [8].

6 Measurement of mixed states and accessible information

An important issue of quantum information theory is the maximum amount of information that one can extract about a quantum system by performing a measurement. Let us consider a simple derivation of this quantity based on conditional and mutual quantum entropies and on simple relations between them. This derivation, akin to the proof of Schumacher *et al.* [22], relies on our information-theoretic description of unitary measurement and does not involve any "environmental" degrees of freedom (it does not involve decoherence induced by an environment [30]). As emphasized before, the entropy that appears in the ancilla A is "extracted" from the measured quantum system Q, whose conditional quantum entropy therefore becomes negative. This means that the quantum system and the ancilla are entangled as a result of the measurement, and that the measurement simply becomes the "act" of ignoring – or tracing over – the quantum system Q which is aimed to be measured. This is in contrast with the prevalent view of measurement, where the quantum system and the ancilla become classically correlated because one is compelled to ignore the numerous degrees of freedom of an uncontrollable environment (in other words, decoherence leads to the selection of a "preferred basis" for the "pointer variable" [30]). As stressed in Section 4, the appearance of a collapse of the wave function can be fully understood when considering subsequent measurements *without* any environment; the statistics of the state of the ancillae that interact sequentially with the quantum system Q is indistinguishable from the statistics resulting from the collapse postulate.

A striking consequence of this information-theoretic interpretation is that, in any measurement of a *pure* state, no information at all (in the sense of Shannon theory) can possibly be extracted from the system. In other words, no information is gained about the identity of the pure state. (This means that the "pointer variable" is *not* classically correlated with the quantum system.) Recognizing that a pure state has a vanishing von Neumann entropy, this turns out to be an obvious result: there is no uncertainty about it, so nothing can be learned from it. This can also be understood as a consequence of the quantum non-cloning theorem [27]: one cannot "clone" (*i.e.*, correlate in the Shannon sense) an arbitrary state with an ancilla, as only entanglement results from the measurement. It is also straightforward to see, by looking at quantum entropies, that the correlations that appear in a measurement do not concern Q *vs.* A but rather concern all the pieces of the (generally macroscopic) ancilla:

the ancilla is "self-consistent" [7]. Clearly, as far as information extraction is concerned, a more interesting case to consider is the measurement of a quantum system Q initially prepared in a *mixed* state; only then can information be extracted about the *preparation* of the state.

A measurement performed on a quantum system which can be prepared in different states yields an amount of information about the preparation which is limited by the Kholevo bound [15]. More precisely, if a system is prepared in a state described by one of the density operators ρ_i $(i = 1, \cdots n)$, with probability p_i, then the information I that can be gathered about the identity of the state is always lower than the Kholevo bound

$$I \leq S(\sum_i p_i \rho_i) - \sum_i p_i S(\rho_i) \ . \tag{46}$$

This result holds for any measurement one can perform on the system, including positive-operator-valued measures (POVM's). Since the original conjecture by Kholevo, a lot of effort has been devoted to obtaining a more rigorous proof of the theorem, or to derivations of stronger upper or lower bounds on I [18,20,28,14,12,22]. Our aim here is to give a simple proof of this upper bound on accessible information which is based on quantum entropies, as opposed to deriving Shannon entropies from the quantum probabilities associated with measurements, as is usually done. The derivation relies only on the unitarity of the measurement seen as a physical process, along with the strong subadditivity property of quantum entropies (*cf.* Sect. 3). This makes the physical content of the Kholevo theorem more transparent: in short, it states that the *classical* mutual entropy (i.e. the acquired information I) is bounded from above by a *quantum* mutual entropy.

Let us assume that we have a "preparer", described by a (discrete) internal variable X, which is distributed according to the probability distribution p_i $(i = 1, \cdots N)$. The internal state of the preparer, considered as a physical quantum system, is given by the density matrix [8]

$$\rho_X = \sum_i p_i |x_i\rangle\langle x_i| \tag{47}$$

with the $|x_i\rangle$ being an orthonormal set of states. The state of the quantum variable X can be copied to another system simply by making conditional

[7] If A_1 and A_2 represent two halves (arbitrarily chosen) of the ancilla, the ternary mutual entropy $S(A_1{:}A_2{:}Q)$ vanishes if the quantum system Q is initially in a pure state and if the measurement process is unitary. But, the ancilla is "self-consistent" that is $S(A_2|A_1) = S(A_1|A_2) = 0$

[8] Of course, ρ_X can be seen as resulting from the partial trace of a pure state in an extended Hilbert space (it can be "purified" via a Schmidt decomposition).

dynamics (the simplest example being a controlled-NOT quantum gate) and in that sense, it behaves like a classical variable (it can be "cloned"). Let us therefore denote by X the collective set of correlated internal variables describing the preparer state. Assume now that the preparer has at his disposal a set of N mixed states ρ_i, and that he chooses one of them for Q according to his internal state X. The joint state of the preparer and the quantum system Q is then given by

$$\rho_{XQ} = \sum_i p_i |x_i\rangle\langle x_i| \otimes \rho_i \tag{48}$$

and a partial trace over X simply gives the state of Q:

$$\rho_Q = \mathrm{Tr}_X \rho_{XQ} = \sum_i p_i \rho_i \equiv \rho \; . \tag{49}$$

The quantum entropy of X, Q and the joint system XQ is given by

$$\begin{aligned} S(X) &= H[p_i] \; , \\ S(Q) &= S(\rho) \; , \\ S(XQ) &= H[p_i] + \sum_i p_i S(\rho_i) \; , \end{aligned} \tag{50}$$

where the last expression results from the fact that ρ_{XQ} is block-diagonal (it is the quantum analogue of the "grouping theorem" in Shannon theory [2]).

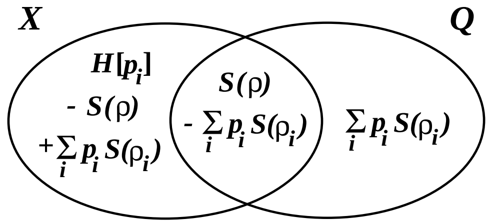
Now, the quantum system Q is "measured" by interacting unitarily with an ancilla A, according to

$$\rho_{X'Q'A'} = (1_X \otimes U_{QA})(\rho_{XQ} \otimes |0\rangle\langle 0|)(1_X \otimes U_{QA})^\dagger \tag{51}$$

where $|0\rangle\langle 0|$ denotes an initial reference state of the ancilla, and X', Q', and A' correspond to the respective systems *after* the unitary evolution U_{QA}. For the moment, let us assume that U_{QA} is arbitrary. The interesting question will be to determine the mutual quantum entropy $S(X'{:}A')$ between the physical state of the ancilla A *after* measurement and the physical state of the preparer X (which remains unchanged in the measurement). We will show that, given certain assumptions for U_{QA}, $S(X'{:}A')$ represents simply the Shannon mutual entropy between the preparer and the ancilla, or, in other words, the information I extracted by the observer about the preparer state.

The relations between the entropies of X and Q before measurement can be summarized by the quantum entropy diagram in Fig. 5. It is easy to calculate

Fig. 5. Entropy Venn diagram for the correlated system XQ before measurement.



the quantum mutual entropy (or mutual entanglement) between X and Q before measurement,

$$S(X{:}Q) = S(X) + S(Q) - S(XQ) = S(\rho) - \sum_i p_i S(\rho_i) \tag{52}$$

showing that $S(X{:}Q)$ is simply the Kholevo bound [see Eq. (46)]. Invoking the upper and lower bounds for the entropy of a convex combination of density matrices (see *e.g.* [25]), *i.e.*,

$$\sum_i p_i S(\rho_i) \leq S\left(\sum_i p_i \rho_i\right) \leq H[p] + \sum_i p_i S(\rho_i) \tag{53}$$

implies

$$0 \leq S(X{:}Q) \leq H[p_i]\ . \tag{54}$$

This shows that the entropy diagram for XQ (represented in Fig. 5) has only positive entries and therefore looks like a classical diagram for correlated variables [9] .

Before measurement, the ancilla A is in a pure state $|0\rangle$ and the joint state of the system XQA is a product state $\rho_{XQ} \otimes |0\rangle\langle 0|$, so that we have $S(X{:}Q) = S(X{:}QA)$. As the measurement involves unitary evolution of QA and leaves X unchanged, it is straightforward to check that this mutual entropy is conserved:

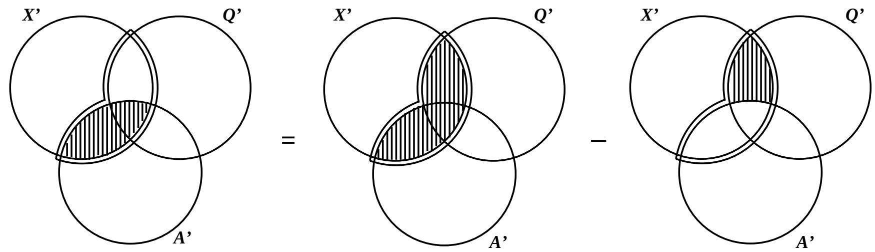
$$S(X'{:}Q'A') = S(X{:}QA) = S(X{:}Q)\ . \tag{55}$$

Next, we may split this entropy according to the quantum analogue of the chain rules for mutual entropies [Eq. (27)] to obtain

$$S(X'{:}Q'A') = S(X'{:}A') + S(X'{:}Q'|A') \tag{56}$$

[9] As explained earlier, this property is related to the fact that ρ_{XQ} is a separable state and therefore is associated with positive conditional entropies.

Fig. 6. Diagrammatic representation of the Kholevo theorem. The area enclosed by the double solid lines represents the mutual entropy that is conserved in the measurement $S(X':Q'A') = S(X:Q)$.



where the second term on the right-hand side is a quantum conditional mutual entropy (*i.e.*, the mutual entropy between X' and Q', conditionally on A'). Combining Eqs. (55) and (56) gives the basic relation

$$S(X':A') = S(X:Q) - S(X':Q'|A') \ . \tag{57}$$

This equation is represented as arithmetic on Venn diagrams in Fig. 6.

Thus, the quantum mutual entropy between the state of the preparer X' (we can ignore the prime since X is unchanged in the measurement) and the state of the ancilla after measurement A' is given by $S(X:Q)$, the Kholevo bound, reduced by an amount which represents the mutual entropy still existing between the preparer's internal variable X and the quantum state after measurement Q', *conditional* on the observed state of the ancilla A'. Since $S(X':Q'|A')$ is in general difficult to calculate, we can make use of strong subadditivity [10] in order to obtain an inequality. In particular, we have $S(X':Q'|A') \geq 0$, which yields the simple upper bound:

$$S(X':A') \leq S(X:Q) = S(\rho) - \sum_i p_i S(\rho_i) \ . \tag{58}$$

It remains to show that, for a particular U_{QA} which describes a measurement, the quantum mutual entropy $S(Q':A')$ reduces to a Shannon mutual entropy (the mutual information I between the state of the preparer and the outcome of the measurement).

Let us consider only the case of a von Neumann measurement [11], using the

[10] Expressed in our quantum information-theoretic language, strong subadditivity implies that the conditional mutual entropy $S(X:Y|Z)$ between any three quantum variables X, Y, and Z is non-negative. This expresses the intuitive idea that the mutual entanglement between X and YZ is larger or equal to the mutual entanglement between X and Z only (just as mutual informations in Shannon theory), so that a mutual entanglement can never decrease when extending a system.

[11] It can be shown that the same reasoning applies also to a positive-operator-valued

explicit form

$$U_{QA} = \sum_\alpha P_\alpha \otimes V_\alpha \tag{59}$$

where the index α refers to the outcome of the measurement and the P_α's denote the projectors in the Q space associated with the measurement ($\sum_\alpha P_\alpha = 1$). The unitary operators V_α act in the A space, and move the ancilla from the initial state $|0\rangle$ to a state $|\alpha\rangle = V_\alpha|0\rangle$ that points to the outcome of the measurement. Let us assume that the $|\alpha\rangle$ are orthogonal so that the outcomes are perfectly distinguishable. The joint density matrix after unitary evolution is thus given by

$$\rho_{X'Q'A'} = \sum_{i,\alpha,\alpha'} p_i |x_i\rangle\langle x_i| \otimes P_\alpha \rho_i P_{\alpha'} \otimes |\alpha\rangle\langle\alpha'| \, . \tag{60}$$

As before, we now have to trace over the quantum system Q' in order to induce correlations between X' and A'. The corresponding density matrix is

$$\rho_{X'A'} = \sum_{i,\alpha} p_i \mathrm{Tr}(P_\alpha \rho_i) |x_i\rangle\langle x_i| \otimes |\alpha\rangle\langle\alpha| \, . \tag{61}$$

As it is a *diagonal* matrix, the relations between the entropies of X' and A' can be described within Shannon theory (the quantum definitions of conditional and mutual entropies reduce to the classical ones in this case.) A simple calculation shows that one has indeed

$$\begin{aligned} S(X'{:}A') &= H\left[\mathrm{Tr}(P_\alpha \rho)\right] - \sum_i p_i H\left[\mathrm{Tr}(P_\alpha \rho_i)\right] \\ &= H(A) - H(A|X) = H(A{:}X) \, , \end{aligned} \tag{62}$$

where $\mathrm{Tr}(P_\alpha \rho_i)$ is the conditional probability $p_{\alpha|i}$ of measuring outcome α on states ρ_i, so that it is justified to identify $S(X'{:}A')$ with the information I in this case. Note that the information gained in the measurement is not described as a difference between initial and final uncertainty of the observer (involving a calculation of probabilities as it is usually done), but rather as a quantum mutual entropy. As a result of Eq. (62), we see that Eq. (58) provides an upper bound on the accessible information, and this completes our derivation of the Kholevo theorem. As shown elsewhere, the same reasoning can be extended to the case of sequential measurements of a quantum system, using chain rules for quantum entropies, providing a generalization of the Kholevo theorem.

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measure (POVM) in general.

As a final remark, let us mention that inequality (58) can be shown to be a special case of a more general relation. For an arbitrary density matrix ρ_{XY} describing a bipartite quantum system whose components interact with ancillae A and B that define bases $|x\rangle$ and $|y\rangle$ respectively, we have clearly $S(A'{:}B') = H(X{:}Y)$, where $H(X{:}Y)$ is the Shannon mutual entropy of the joint probability $p_{xy} = \langle x, y|\rho_{XY}|x, y\rangle$. Using

$$\begin{aligned} S(X{:}Y) &= S(X'A'{:}Y'B') \\ &= S(A'{:}B') + S(A'{:}Y'|B') + S(X'{:}Y'B'|A') \end{aligned} \tag{63}$$

and the non-negativity of conditional mutual entropies yields the general inequality

$$H(X{:}Y) \leq S(X{:}Y) \tag{64}$$

between classical and quantum mutual entropies.

7 Conclusions

We have shown that quantum entanglement can be consistently described using the notion of negative conditional entropy, an essential feature of a quantum information theory built entirely on density matrices. Negative quantum entropy can be traced back to "conditional" density matrices which admit eigenvalues larger than unity. A straightforward definition of quantum mutual entropy, or mutual entanglement, can also be obtained using a "mutual" density matrix. This quantum matrix-based formalism gives rise to the violation of well-known bounds in classical information theory. It treats quantum entanglement and classical correlation on the same footing, while clarifying in which sense entanglement can induce correlation. This last point allows for a consistent information-theoretic description of unitary quantum measurement, devoid of any assumption of wave-function collapse, which, at the same time, accounts for the creation of entropy (random numbers) in the measurement outcome. This sheds new light for example on information-theoretic aspects of Bell-type experiments or on the issue of how much information can be accessed in a quantum measurement. Also, as quantum entanglement is a central feature of quantum computation, we believe that the present formalism will shed new light on decoherence (entanglement with an environment) in noisy quantum channels, as well as the error-correcting codes being devised to counteract it. From a more fundamental point of view, the fact that quantum conditional entropies can be negative reveals that quantum statistical mechanics is qualitatively very different from classical statistical mechanics, even though most of the formulae are similar.

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